ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Information and Computation 204 (2006) 610–636

Information and Computation

www.elsevier.com/locate/ic

Final coalgebras for functors on measurable spaces

Lawrence S. Moss, Ignacio D. Viglizzo *

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

Received 20 July 2004; revised 05 January 2005
Available online 13 February 2006

Abstract

We prove that every functor on the category Meas of measurable spaces built from the identity and constant functors using products, coproducts, and the probability measure functor Δ has a final coalgebra. Our work builds on the construction of the universal Harsanyi type spaces by Heifetz and Samet and papers by Rößiger and Jacobs on coalgebraic modal logic. We construct logical languages, probabilistic logics of transition systems, and interpret them on coalgebras. The final coalgebra is carried by the set of descriptions of all points in all coalgebras. For the category Set, we work with the functor $\mathcal{D}$ of discrete probability measures. We prove that every functor on Set built from $\mathcal{D}$ and the expected functors has a final coalgebra. The work for Set differs from the work for Meas: negation is needed for final coalgebras on Set but not for Meas.

Keywords: Coalgebra; Final coalgebra; Measurable space; Probability measure; Modal logic

1. Introduction

One of the themes in coalgebra the last few years has been the search for *solution principles going beyond finality*. The idea is that the finality principle, while interesting and useful, has two limitations. First of all, one has to know that final coalgebras exist for various functors on various categories.

* Corresponding author. Present address: Departamento de Matemática, Universidad Nacional del Sur, Avda. Alem 1253, 2do piso, (8000) Bahía Blanca, Argentina. Fax: +54 291 459 5162.
E-mail addresses: lsm@cs.indiana.edu (L.S. Moss), viglizzo@criba.edu.ar (I.D. Viglizzo).

0890-5401/$ - see front matter
doi:10.1016/j.ic.2005.04.006

This has been most thoroughly investigated in the case of sets; by now there are several sufficient conditions which imply the existence of final coalgebras for most if not all of the functors of current interest. And second, even granted the existence of final coalgebras, one often wants to define and study functions which are not literally coalgebra maps into the final coalgebra. This concern shows up in many places, including work on functional programming using coalgebra and also coalgebraic treatments of the classical area of recursive program schemes. Here one would like to see whether coalgebra can apply to issues at the heart of the semantics of computation. The jury is still out on whether this will turn out to be useful, but there is already a treatment of uninterpreted and interpreted recursive program schemes that calls on a large body of work in coalgebraic recursion theory. We shall have nothing to say about this in this paper. But we do have a contribution to the first matter we mentioned, the matter of *solving equations in structured categories*. Here there is a parallel to what one does in domain theory when one looks for solutions to equations like $D = D \to D$. One wants to solve interesting equations in categories such as topological spaces. We are not concerned with this equation, but rather with "tamer" ones.

Our main theorem concerns the category Meas of measurable spaces. We show that every functor on Meas which is built from the identity and constant functors using products, coproducts, and the probability measure functor Δ that assigns to each measurable space the set of all probability measures on it, with a measurable space structure. For example, we are interested in the solution of equations like

$$X = M_1 \times \Delta(M_2 \times X). \tag{1.1}$$

Here M_1 and M_2 are fixed measurable spaces. The equal sign "=" in (1.1) may be read as "isomorphism" instead of literal equality. In general, such an equation has many solutions; for example the empty space solves (1.1) trivially. But one is usually interested in the "largest" solution.

Here is how we formalize this. Consider an equation $X = T(X)$ as in (1.1). A *coalgebra* of T is a pair (A, f) consisting of a measurable space A and a measurable map $f : A \to TA$. Notice that one very special kind of coalgebra would be one where f were an isomorphism. If (A, f) and (B, g) are coalgebras of T, then a *coalgebra morphism* from (A, f) to (B, g) is a measurable $i : A \to B$ such that $g \circ i = Ti \circ f$. (A, f) is *final* if for every (B, g) there is exactly one coalgebra morphism from (B, g) to (A, f). A well-known lemma of Lambek, valid not only for measurable spaces but in any category whatsoever, states that if (A, f) is final, then f must be an isomorphism (in the categorical sense, hence for measurable spaces f will be a measurable bijection with measurable inverse). The point is that a final coalgebra for T, if it exists, will automatically be a solution to $X \cong T(X)$. And the finality condition insures that final coalgebras A for T are "universal" in the sense that every coalgebra, in particular every space B isomorphic to TB would sit inside A in a unique way. So in this way, we pursue final coalgebras as a way to solve equations like (1.1) and at the same time obtain universal objects.

Once again, our result is that for an important family of functors T, a final coalgebra exists. This final coalgebra is the "biggest" possible coalgebra, and it is of interest because it contains a unique copy (up to bisimulation) of every coalgebra.

The finality result is perhaps surprising, since one often assumes that Δ is an analog of the power set functor $\mathcal{P}$ on Set. $\mathcal{P}$ has no final coalgebra on Set. (It does have one on the related category of classes.) In any case, the methods for proving the final coalgebra results on Meas and Set are different. We indicate in Section 7 that one could use our methods on sets. The result here would not be new, but perhaps the proof will be of interest. The final coalgebra result for Meas is not only interesting in and of itself, it also permits application of the growing body of general results from coalgebra.

There is an "ideological" aspect of our work that is worth mentioning as well. It is often remarked that coalgebra is about *observation* rather than *construction*. One way to give this slogan some content is to show that final coalgebras may be represented in terms of logics with an "observational" flavor. These are modal logics, or some generalization of them. One other theme of the work in coalgebra in the past five or ten years has been proposals for, and studies of, versions of modal logic for coalgebra. In this paper, we adapt the version of *coalgebraic modal logic* proposed by Rößiger in his papers [1,2] and then developed further by Jacobs [3]. Indeed, we follow Jacobs' work and notation. But we must adapt these from Set to Meas, and so there are some differences. In any case, we construct final coalgebras from the theories (i.e., sets of formulas) satisfied by all the points in all possible coalgebras. The main difficulty is to endow this collection with the appropriate structures. For this, we had to ferret out some results from measure theory.

There is a predecessor to this kind of work. We have in mind the result of Kupke, Kurz, and Venema [4] that every functor on the category of Stone spaces built from the identity, constant functors, products and coproducts, and the *Vietoris endofunctor* has a final coalgebra. Their proof uses an algebraic construction and duality, in contrast to our more logic-based efforts. Of course it would be interesting to have several proofs for these kinds of finality results.

Another motivation for our work is that special cases of our main result are already known. In the literature on game theory and economics, there exists a long discussion of *universal type spaces*. Type spaces were introduced in Harsanyi [5]. They are mathematical structures used in modeling settings where agents are described by their *types*, and these types give us "beliefs about the world," "beliefs about each other's beliefs about the world," "beliefs about each other's beliefs about each other's beliefs about the world," etc. That is, the formal concept of a type space is intended to capture in one structure an unfolding infinite hierarchy related to *interactive belief*. Harsanyi did not really formalize type spaces in his original paper; this was left to later researchers starting with Böge and Eisele [6]. The constructions of universal type spaces in that paper and most of the succeeding literature worked on categories which combined measure-theoretic and topological structure. The topological work is to some extent unfortunate, since it does not appear to be close to the original motivations for type spaces. Heifetz and Samet's paper [7] was the first to avoid the topological setting, and it seems to have been written explicitly for the purpose of getting away from the extra topological assumptions that had been common in the area. (Incidentally, it will be interesting to adapt and extend our work to the settings that contain both topological and measure-theoretic notions.) We are influenced by this paper both because we stick to measurable spaces and also because we generalize their logical system to a wide class of functors on measurable spaces. The main influence is that we retain the overall architecture of their first proof of the existence of universal type spaces. But type spaces per se are not exactly the coalgebras of any functor on measurable spaces; there are side conditions that add complications.[1]

[1] Harsanyi type spaces are close to coalgebras of $\Delta(M \times Id)$ for a fixed M. But type spaces are about several "players" or agents, so they really are coalgebras of a related functor on a category Meas^I for a fixed set I. And then one imposes some conditions on the functor related to the intuitions of self-knowledge by the players. We discuss coalgebraic modeling of Harsanyi type spaces in detail in our earlier paper [8].

In addition to the application from theoretical economics, the matter of final coalgebras for functors on measurable spaces has been considered in the theoretical computer science literature; cf. [9,10]. Our final coalgebra theorem does not appear in these papers, and our methods also are different. The papers [9,10] also explore related matters which we do not study, such as the interaction with recursion, model checking, and the like. Again, these would be interesting extensions of our work.

2. Preliminaries on measurable spaces

A *boolean algebra* of sets is a family of sets closed under complement and union. If the family is in addition closed under countable unions, then it is a σ-*algebra*. A *measurable space* is a pair $M = (M, \Sigma)$, where M is a set and Σ is a σ-algebra of subsets of M. The sets in Σ are called *measurable sets* or *events*. Usually Σ contains all singletons $\{x\}$. (For example, the Borel algebra in any T_0 topological space, the smallest σ-algebra containing the open sets, has this property.) Even more usually, one expects a weaker condition that for each $x \in M$, $\{x\}$ is the intersection of the measurable subsets of M containing x. However, our work applies to all measurable spaces, including ones which do not satisfy this condition. A collection $\mathcal{B}$ of subsets of M *generates* a σ-algebra Σ if Σ is the smallest σ-algebra including $\mathcal{B}$. A π-*system* is a class $\mathcal{A}$ of subsets of M closed under the formation of finite intersections. A *measure on M* is a σ-additive function $\mu : \Sigma \to [0, \infty]$ such that $\mu(\emptyset) = 0$. The measure μ is a *probability measure* if $\mu(M) = 1$.

A morphism of measurable spaces $f : (M, \Sigma) \to (N, \Sigma')$ is a function $f : M \to N$ such that for each $A \in \Sigma'$, $f^{-1}(A) \in \Sigma$. This gives a category which is often called Meas. Meas has products and coproducts; indeed it has all limits (but we do not need this). There is an endofunctor $\Delta : \mathsf{Meas} \to \mathsf{Meas}$ defined by: $\Delta(M)$ is the set of probability measures on M with the σ-algebra Σ_Δ generated by $\{\beta^p(E) \mid p \in [0, 1], E \in \Sigma\}$, where

$$\beta^p(E) = \{\mu \in \Delta(M) \mid \mu(E) \geqslant p\}.$$

Here is how Δ acts on morphisms. If $f : M \to N$ is measurable, then for $\mu \in \Delta(M)$ and $A \in \Sigma'$, $(\Delta f)(\mu)(A) = \mu(f^{-1}(A))$. That is, $(\Delta f)(\mu) = \mu \circ f^{-1}$. To prove that Δf is measurable we need the following Lemma.

Lemma 2.1. *For each $p \in [0, 1]$, β^p may be regarded as a "predicate lifting." That is β^p takes measurable subsets of each space M to measurable subsets of $\Delta(M)$, and it is natural in the sense that if $f : M \to N$, then for all measurable $E \subseteq N$, $\beta^p(f^{-1}(E)) = (\Delta f)^{-1}(\beta^p(E))$.*

Proof. Both sets are $\{\mu \in \Delta M \mid \mu(f^{-1}(E)) \geqslant p\}$. $\square$

We also note some additional structure. First, there is a natural transformation $\delta : Id \to \Delta$ defined by $\delta_M(m)(E) = 1$ if $m \in E$ and 0 if $m \notin E$. We also write δ_m instead of $\delta_M(m)$; this is the Dirac measure supported at m. Second, there is a natural transformation $\gamma : \Delta\Delta \to \Delta$ given by $\gamma_M(\mu)(E) = \int_{\nu \in \Delta(M)} \nu(E)\, \mathrm{d}\mu$.

Lemma 2.2 (Giry [11]). *(Δ, δ, γ) is a monad on* Meas.

We won't really use this result, but we do need to know that Δ is a functor. For applications in coalgebra one often prefers to work with functors which preserve weak pullbacks. However, it turns out that Δ does not have this property (see [12]).

Lemma 2.3 (Heifetz and Samet [7], Viglizzo [13]). *Let $\mathcal{A}$ be a π-system of sets which generates the σ-algebra Σ on a measurable space M. Then the family of sets*

$$\{\beta^p(E) \mid E \in \mathcal{A} \text{ and } p \in [0,1]\}$$

generates the σ-algebra Σ_Δ.

Lemma 2.3 was first proved in [7] in the case when $\mathcal{A}$ is a boolean algebra of sets. The extension to π-systems is due to Viglizzo [13]; we need the stronger result so we can use Dynkin's π–λ Theorem, as does the next result.

Theorem 2.4. *Suppose that μ_1 and μ_2 are probability measures on $\sigma(\mathcal{A})$, where $\mathcal{A}$ is a π-system and $\sigma(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$. If μ_1 and μ_2 agree on $\mathcal{A}$, then they agree on $\sigma(\mathcal{A})$.*

For more details on Theorem 2.4, see, for example, Billingsley [14], p. 36.

2.1. The probability measure functor Δ does not preserve ω^{op} limits

One standard approach for finding final coalgebras for a functor T on some category goes via the final sequence of T. The final sequence is given by

$$1 \xleftarrow{\;!\;} T1 \xleftarrow{\;T!\;} T^2 1 \xleftarrow{\;T^2!\;} \cdots \xleftarrow{\;T^{n-1}!\;} T^n 1 \xleftarrow{\;T^n!\;} T^{n+1}1 \qquad \cdots$$

Actually, the sequence might go on transfinitely, but let us assume for a moment that the first ω steps have a limit $T^\omega 1$ with maps $f_n : T^\omega 1 \to T^n 1$. We make the further critical assumption that T preserves this limit. Then the universal property of the limit gives a map $g : T^\omega \to T^{\omega+1}1 = T(T^\omega 1)$, and this equips $T^\omega 1$ with the structure map of a final coalgebra for T.

One would therefore like to use this idea in the case of functors on Meas such as Δ. (As it happens, for Δ itself, we have already that $\Delta 1 \cong 1$. So the final sequence gives a final coalgebra for a trivial reason. But we shall be interested in more complicated functors such as $[0,1] \times \Delta X$ or $[0,1] + \Delta X$. We call the functors of interest the *measure polynomial functors*, see Section 3.) However, there is a counterexample to the the critical assumption above.

Proposition 2.5. *The functor Δ does not preserve ω^{op} limits.*

For details on this, see Viglizzo [13]. The explicit counterexample is based on work of Andersen and Jessen [15] (see also Halmos [16]). The same negative result holds for more complicated functors, though we shall not show this.

One can imagine several ways to use the final sequence despite the result in Proposition 2.5. (1) Perhaps each functor T of interest preserves the limit $T^\omega 1$ of the first ω steps of its final sequence. (We only know that there are *some* ω^{op} limits which are not preserved.) Our hunch is that for functors T involving Δ, $T^\omega 1$ will not be preserved and hence will not be the carrier of a final coalgebra. (2) The final sequence iterates into the transfinite in a canonical way. As James Worrell has shown in [17], for a finitary functor T on Set, $T^{\omega\cdot 2}1$ is the carrier of a final coalgebra. This accounts for the final coalgebras on Set that come up in practice, so it would be interesting to adapt his results in our setting. But much of his work uses features of Set that are false of Meas. We have no results on the final sequences in Meas, but results in this direction would obviously be of independent interest. (3) One might try to use the final sequence in a more sophisticated way than simply taking its limit. We discuss some work in this direction in Section 2.3 below. (4) The economics/game theory literature goes in a different direction than all of this, by moving to a category defined by some topological condition on the spaces, and with functors which do preserve ω^{op} limits. Concrete examples include the category of *Hausdorff* topological spaces; here ΔX is the set of all *regular Borel* measures on X. Kolmogorov's consistency theorem may be applied to get the preservation of the limits. This is essentially how *universal type spaces* were obtained in several papers (see, for example [18,19]). Since we want to prove the final coalgebra result in full generality without restricting to a "nice" class of measures, this is not our approach. (5) Using domain theory, van Bruegel et al. prove in [20] the existence of a final coalgebra for a related functor M that assigns to each measurable space the space of all *subprobability* measures. Maybe this approach could be adapted to the functors we deal with in this paper, but again, this is not the road we have taken.

2.2. The main idea in our construction

To get final coalgebras, we adapt *coalgebraic modal logic*. The functors we are interested in will be called *measurable polynomial functors*. They are the functors built from the identity functor Id and the constant functors, and closed in the following ways: if U and V are measure polynomials, then so are $U + V$, $U \times V$, and ΔU. We fix such a functor T in this discussion. For T, we shall construct a coalgebraic modal language $\mathcal{L}(T)$. (Incidentally, $\mathcal{L}(T)$ is simply a language with a syntax and a semantics. In this paper we do not propose any logical system for validity.) $\mathcal{L}(T)$ will be sorted. ("Sorts" are usually called "types," but we avoid this word due to the connection with Harsanyi type spaces.) In this discussion, we use S for an arbitrary "ingredient" of T: a functor used in building T, or the identity Id. T will always be an ingredient of itself. The sorts are the ingredients of T, and we write $\varphi : S$ to say that φ is of sort S. It will turn out that the formulas of sort Id will not be trivial; indeed, the syntax of $\mathcal{L}(T)$ is constructed so that the formulas of sort T are subformulas of those of sort Id. A formula $\varphi : S$ may be interpreted in a T-coalgebra (X, c). Our semantics gives subsets $[\![\varphi]\!]^c_S \subseteq SX$; these are the points of SX satisfying φ.

Turning things around, we can associate with each $x \in SX$ its *S-description* $d^c_S(x)$, this is the set of $\varphi : S$ satisfied by x. The leading idea in our work is to consider the set Id^* of Id-descriptions of all points in all coalgebras (X, c), and more generally the set S^* of all S-descriptions of all points in all SX. We equip each S^* with a σ-algebra obtained canonically from the "measurable" formulas in the language, which are denoted as $\varphi :: S$. That is, for $\varphi :: S$, let $|\varphi|_S = \{s \in S^* \mid \varphi \in s\}$. The family of sets of the form $|\varphi|$ generates a σ-algebra, and in this way S^* is a measurable space. Indeed, since $\mathcal{L}(T)$ has classical conjunction on all sorts, this family is a π-system.

We also take a carefully constructed coalgebra structure $c^* : Id^* \to T^*$. The unique map from a coalgebra (X, c) to (Id^*, c^*) turns out to be d^c_{Id}. This is a pleasing result because final coalgebras are thereby built from descriptions of points in coalgebras. In any case, to carry out the leading idea one has to be careful in formulating the language $\mathcal{L}(T)$. It turns out that there is a canonical map from Id^* to T^*, and then c^* is the composition of this map with a special map $r_T : T^* \to T(Id^*)$. More generally, for each ingredient S we need a canonical map $r_S : S^* \to S(Id^*)$. The measure-theoretic work that we mentioned in Lemma 2.3 and Theorem 2.4 comes into a result which is ultimately used in building r_T and hence c^*. Specifically, we need for each ingredient ΔS of T a map $\epsilon : (\Delta S)^* \to \Delta(S^*)$. And this means that we'll need to define probability measures on spaces of the form S^*. For this, it will be crucial to work with a set of generators of the space $\Delta(S^*)$. Our overall language $\mathcal{L}(T)$ is "tuned" so that we have a π-system of generators of the space $\Delta(S^*)$: this will be the family of sets of the form $\beta^p|\varphi|$ for $\varphi :: S$. Now we can see a reason to include modal operators corresponding to the β^p operations. On the other hand, negation is not needed in $\mathcal{L}(T)$ due to the fact that a probability measure is uniquely determined by its actions on a π-system of generators.

2.3. Doing without the logic

After writing this paper, the second author pursued a different direction concerning final coalgebras for endofunctors T on Meas. He has shown (see Viglizzo [12,13]) the existence of a final coalgebra for T in the following way. Consider $T^\omega 1$. Each coalgebra maps canonically into each object of the final sequence of T, and so we may consider the set Z of all images of all points in all coalgebras. It turns out that Z is the carrier of a final coalgebra structure. Also, the approach here is closer to the "ideological" point that final coalgebras may be considered as the records of all possible "observations," where "observations" here is formalized in terms of some version of modal logic.

3. Syntax and semantics

Definition 3.1. The class of *measure polynomial functors* is the smallest class of functors on Meas containing the identity functor Id, the constant functor M for each measurable space M and closed in the following ways: if U and V are measure polynomials, then so are $U + V$, $U \times V$, and ΔU. (In effect, we are specifying a syntax for the functors of interest.)

For a measure polynomial functor T, we define a finite set $\mathsf{Ing}(T)$ of functors by the following recursion: For the identity functor, $\mathsf{Ing}(Id) = \{Id\}$; for a constant space M, $\mathsf{Ing}(M) = \{M, Id\}$, $\mathsf{Ing}(U \times V) = \{U \times V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$, and similarly for $U + V$; $\mathsf{Ing}(\Delta S) = \{\Delta S\} \cup \mathsf{Ing}(S)$. We call $\mathsf{Ing}(T)$ the set of *ingredients of* T. Our definition ensures that the identity is always an ingredient of T, even if T is a constant functor. Each measure polynomial functor T has only finitely many ingredients.

Example 3.2. Let $[0,1]$ be the unit interval of the reals, endowed with the usual Borel σ-algebra, and $T = [0,1] \times (\Delta Id + \Delta Id)$. Then

$$\mathsf{Ing}(T) = \{Id, [0,1], \Delta Id, \Delta Id + \Delta Id, [0,1] \times (\Delta Id + \Delta Id)\}.$$

Of course, $[0,1]$ here is the constant functor on Meas whose value is the space $[0,1]$. Also, notice that we take the ingredients to be a *set*; there is no need to have two copies of ΔId to match the two occurrences of this functor inside the syntactic expression for T. To see how some of our notation works, let $U = \Delta Id$ and let $V = \Delta Id + \Delta Id$. Then we have $V = U + U$ and $T = [0,1] \times V$.

$$\mathsf{true}_S : S \qquad \frac{A \subseteq M \text{ measurable or a singleton}}{A : M}$$

$$\frac{\varphi : S \quad \psi : S}{\varphi \wedge \psi : S} \qquad \frac{\varphi : U \quad \psi : V}{\langle \varphi, \psi \rangle_{U \times V} : U \times V}$$

$$\frac{\varphi : U}{\mathsf{inl}_{U+V}\varphi : U + V} \qquad \frac{\varphi : V}{\mathsf{inr}_{U+V}\varphi : U + V}$$

$$\frac{\varphi :: S, p \in [0,1]}{\beta^p \varphi : \Delta S} \qquad \frac{\varphi : T}{[\mathsf{next}]\varphi : Id}$$

Fig. 1. The syntax of $\mathcal{L}(T)$. The notation $\varphi :: S$ in the $\beta^p\varphi$ formation rule means that for every constant functor $M \in \mathsf{Ing}(T)$, every subformula of φ of sort M is a *measurable* set.

Syntax. We define a language $\mathcal{L}(T)$ in Fig. 1. $\mathcal{L}(T)$ is sorted, and there is a sort for each ingredient S of T. We write $\varphi : S$ to say that φ is a formula of sort S.

We say that a formula is of *constant sort* if its sort is a constant functor M. The formulas of $\mathcal{L}(T)$ of sort M are the measurable sets in M together with the singletons (i.e., one element subsets) of M. We shall use A to denote either a measurable subset or a singleton. The reason we add the singletons is that we shall want different elements of M to differ on some sentence. This feature of $\mathcal{L}(T)$ will be used in Lemma 5.1. (Of course, in most spaces the singletons are measurable. In such spaces, some of our definitions simplify.) We further write $\varphi :: S$ to mean that all of the subformulas of φ of constant sort are measurable sets. (So if all constant spaces in $\mathsf{Ing}(T)$ have the property that all singletons are measurable, then $\varphi : S$ implies $\varphi :: S$.)

It is important to note that in formulas $\beta^p\varphi : \Delta S$, we must have $\varphi :: S$. Otherwise, as we shall see below, the semantics would not be well-defined. Also, in our syntax, p may be any real number in the unit interval $[0, 1]$.

The main reason for the formulas true_S is to insure that the language is not empty. For example, if $T = \Delta(M \times Id)$ and we did not have true_{Id}, then $\mathcal{L}(T)$ would be empty. It would be sufficient to only take as primitive true_{Id}, true_M and true_{U+V} for the coproduct spaces; then the other true_S are definable from them: $\mathsf{true}_{U\times V} = \langle \mathsf{true}_U, \mathsf{true}_V \rangle$, $\mathsf{true}_{\Delta S} = \beta^1 \mathsf{true}_S$. We have chosen to have true_S as a primitive for all S; the other choice would have worked as well.

It should be noted that $\mathcal{L}(T)$ includes neither disjunction nor negation. This is because by omitting those connectives we obtain a stronger result. That is, the weaker the logic that does a certain job, the stronger the result. Having said this, there is no problem at all in adding standard connectives to our languages. For many purposes, one would indeed want to do this. But for this paper, there is no reason to go beyond conjunction.

Example 3.3. Let T be as in Example 3.2. Recall that our ingredients were Id, $[0,1]$, $U = \Delta Id$, $V = U + U$, and $T = [0,1] \times V$. Here are some examples of formulas in $\mathcal{L}(T)$, along with their sorts: $\mathsf{true}_{Id} : Id$; $\{1\} : [0,1]$, and indeed $\{1\} :: [0,1]$; $\beta^1\mathsf{true}_{Id} : U$; $\mathsf{inl}\,\beta^1\mathsf{true}_{Id} : V$; $\varphi_1 = \langle \{1\}, \mathsf{inr}\,\beta^1\mathsf{true}_{Id} \rangle : T$; $\varphi_1 \wedge \mathsf{true}_T : T$; $[\mathsf{next}]\varphi_1 : Id$; and $\beta^{1/2}[\mathsf{next}]\varphi_1 : U$.

$$
\begin{array}{rclrcl}
[\![\mathsf{true}]\!]^c_S & = & SX & [\![\mathsf{inl}\,\varphi]\!]^c_{U+V} & = & \mathsf{inl}([\![\varphi]\!]^c_U) \\
[\![A]\!]^c_M & = & A & [\![\mathsf{inr}\,\varphi]\!]^c_{U+V} & = & \mathsf{inr}([\![\varphi]\!]^c_V) \\
[\![\varphi \wedge \psi]\!]^c_S & = & [\![\varphi]\!]^c_S \cap [\![\psi]\!]^c_S & [\![\beta^p\varphi]\!]^c_{\Delta S} & = & \beta^p([\![\varphi]\!]^c_S) \\
[\![\langle\varphi,\psi\rangle]\!]^c_{U\times V} & = & [\![\varphi]\!]^c_U \times [\![\psi]\!]^c_V & [\![[\mathsf{next}]\varphi]\!]^c_{Id} & = & c^{-1}([\![\varphi]\!]^c_T)
\end{array}
$$

Fig. 2. The semantics of $\mathcal{L}(T)$.

Semantics. Let $c : X \to TX$ be a coalgebra of T. The semantics assigns to each $S \in \mathsf{Ing}(T)$ and each $\varphi : S$ a subset $[\![\varphi]\!]^c_S \subseteq SX$. The definition is by recursion on the language $\mathcal{L}(T)$. It is given in Fig. 2.

We check easily that if $\varphi :: S$, then $[\![\varphi]\!]^c_S$ is a measurable subset of SX. (Please note that we need the restriction on subformulas of constant sort for this observation.) So $\beta^p[\![\varphi]\!]$ makes sense. This leads to an inductive proof that our semantics is well-defined. While on the topic of the $\beta^p[\![\varphi]\!]$ formulas, note that

$$\mu([\![\varphi]\!]^c_S) = \max\{p : \mu \in [\![\beta^p\varphi]\!]^c_{\Delta S}\}. \tag{3.1}$$

To understand this, recall that the probability measure μ assigns some real number, say q, to the measurable set $[\![\varphi]\!]^c_S$. For each $p \leqslant q$, we have $\mu([\![\varphi]\!]^c_S) \geqslant p$; for $p > q$, we do not have $\mu([\![\varphi]\!]^c_S) \geqslant p$. Now to say that $\mu([\![\varphi]\!]^c_S) \geqslant p$ is the same as to say $\mu \in [\![\beta^p\varphi]\!]^c_S$. So overall, our q is the largest p such that $\mu \in [\![\beta^p\varphi]\!]^c_S$.

As the reader has noticed, we dropped the superscripts on the pairing and inclusion operators. We shall continue this practice, since those subscripts are usually clear from the context. We also will occasionally omit the superscript c and the sort subscript when dealing with the semantics of a formula $\varphi : S$ on a particular coalgebra $c : X \to TX$.

Example 3.4. We continue developing Examples 3.2 and 3.3. Let $X = \{x, y, z\}$ with the σ-algebra generated by the singletons (so all subsets are measurable). Then $TX = [0,1] \times (\Delta X + \Delta X)$. Since X is so small, we may denote measures on it by triples (q_1, q_2, q_3) of non-negative reals which sum to 1. For example $(1/2, 1/4, 1/4)$ represents a measure in ΔX having the indicated values on x, y, z, respectively. Let $c : X \to TX$ be

$$
\begin{aligned}
c(x) &= \langle 1, \mathsf{inr}(1/3, 1/3, 1/3)\rangle \\
c(y) &= \langle 1/2, \mathsf{inl}(1/2, 1/4, 1/4)\rangle \\
c(z) &= \langle 0, \mathsf{inr}(0, 0, 1)\rangle
\end{aligned}
$$

We give the semantics of the formulas mentioned in Example 3.3.

$$
\begin{aligned}
[\![\mathsf{true}_{Id}]\!]^c_{Id} &= X \\
[\![\{1\}]\!]^c_{[0,1]} &= \{1\} \\
[\![\beta^1\mathsf{true}_{Id}]\!]^c_U &= \Delta X \\
[\![\varphi_1]\!]^c_T &= \{1\} \times \mathsf{inr}(\Delta X) \\
&= [\![\varphi_1 \wedge \mathsf{true}_T]\!]^c_T \\
[\![[\mathsf{next}]\varphi_1]\!]^c_{Id} &= c^{-1}([\![\varphi_1]\!]^c_T) \\
&= \{x\} \\
[\![\beta^{1/2}[\mathsf{next}]\varphi_1]\!]^c_U &= \{(q_1, q_2, q_3) \in \Delta(X) | q_1 \geqslant 1/2\}
\end{aligned}
$$

Example 3.5. We consider the case of the measure polynomial functor ΔId itself. In this case, $\mathsf{Ing}(\Delta Id) = \{\Delta Id, Id\}$. Let $\mathbb{R}$ be the reals with the Lebesgue measure λ. Let $c : \mathbb{R} \to \Delta(\mathbb{R})$ be

$$c(a)(E) = \frac{1}{\sqrt{2\pi}} \int_E \mathrm{e}^{-\frac{(x-a)^2}{2}} \,\mathrm{d}\lambda \,.$$

This integral comes from the normal distribution with $\sigma = 1$ and $\mu = a$. We omit the verification that c is a measurable function on $\mathbb{R}$. One can check that for all $\varphi : Id$, $[\![\varphi]\!]^c_{Id} = \mathbb{R}$, and for all $\varphi : \Delta Id$, $[\![\varphi]\!]^c_{\Delta Id} = \Delta\mathbb{R}$. (The situation is analogous to modal logic without atomic propositions on a model where every world has a successor; all worlds are bisimilar and hence have the same modal theory.)

Example 3.6. Things are more complicated with $T' = \Delta([0,1] + Id)$. We have

$$\mathsf{Ing}(T') = \{Id,\ [0,1],\ [0,1] + Id,\ T'\}.$$

Let's consider a T'-coalgebra, with the same carrier $X = \{x, y, z\}$ from Example 3.4 and with $c(x)(E) = \int_{E \cap \mathsf{inl}[0,1]} 2t \, d\lambda(T), c(y)(E) = \lambda(E) = \int_{E \cap \mathsf{inl}[0,1]} d\lambda$ for all measurable subsets E of $[0,1] + X$, and $c(z) = \lambda/2 + \delta_x/4 + \delta_y/8 + \delta_z/8$.

In this example, $c(x)$ belongs to $[\![\beta^{3/4}\mathsf{inl}[1/2,1]]\!]$, but $c(y)$ does not.

4. The spaces S^*

At this point, we have the syntax and semantics of $\mathcal{L}(T)$. We now turn to the study of "canonical" spaces which turn out to include the carrier of the final coalgebra.

4.1. The description operations d^c_S

Lemma 4.1. *Coalgebra morphisms preserve the semantics. That is, if $f : b \to c$ is a morphism of coalgebras $b : X \to TX$ and $c : Y \to TY$, and if $\varphi : S$, then $(Sf)^{-1}([\![\varphi]\!]^c_S) = [\![\varphi]\!]^b_S$.*

Proof. By induction on $\mathcal{L}(T)$. The base cases of formulas true_S and $A : M$ are trivial: $(Sf)^{-1}([\![\mathsf{true}_S]\!]^c_S) = (Sf)^{-1}(SY) = SX = [\![\mathsf{true}_S]\!]^b_S$, and $(Mf)^{-1}([\![A]\!]^c_S) = (Mf)^{-1}(A) = A = [\![A]\!]^b_S$. Since inverse images preserve intersections, it's also easy to prove the result for conjunctions. We check this for formulas $\beta^p\varphi$ and $[\mathsf{next}]\varphi$ assuming the result for φ. We calculate:

$$\begin{aligned}
(\Delta Sf)^{-1}([\![\beta^p\varphi]\!]^c_{\Delta S}) &= (\Delta Sf)^{-1}(\beta^p[\![\varphi]\!]^c_S) \\
&= \beta^p((Sf)^{-1}([\![\varphi]\!]^c_S)) \\
&= \beta^p[\![\varphi]\!]^b_S \\
&= [\![\beta^p\varphi]\!]^b_{\Delta S}
\end{aligned}$$

We used Lemma 2.1. For $[\mathsf{next}]\varphi : Id$ we have

$$\begin{aligned}
f^{-1}([\![[\mathsf{next}]\varphi]\!]^c_{Id}) &= f^{-1}(c^{-1}([\![\varphi]\!]^c_T)) \\
&= b^{-1}((Tf)^{-1}([\![\varphi]\!]^c_T)) \\
&= b^{-1}([\![\varphi]\!]^b_T) \\
&= [\![[\mathsf{next}]\varphi]\!]^b_{Id}
\end{aligned}$$

Here we used the fact that f is a coalgebra morphism: $c \circ f = Tf \circ b$. The remaining induction steps are similar, indeed easier. $\square$

Definition 4.2. For each coalgebra $c : X \to TX$ and each $x \in SX$, we define

$$d^c_S(x) = \{\varphi : S \mid x \in [\![\varphi]\!]^c_S\}.$$

We call each such set $d^c_S(x)$ a *satisfied theory*.

Definition 4.3. We next define the *canonical sets* S^* for $S \in \mathsf{Ing}(T)$ by

$$S^* = \{d^c_S(x) \mid x \in SX \text{ for some coalgebra } c : X \to TX\}. \tag{4.1}$$

Note that even though there is a proper class of coalgebras for T, each S^* really is a set; indeed it has cardinality at most $2^{\mathfrak{c}\gamma}$, where $\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the continuum, and γ is the maximum of the cardinalities of the sets of points or measurable subsets of the constant functors in $\mathsf{Ing}(T)$. We will use the letter s for elements of S^*.

Definition 4.4. For $\varphi : S$, we define

$$|\varphi|_S = \{s \in S^* \mid \varphi \in s\}. \tag{4.2}$$

Then we have immediately that $\varphi \in d^c_S(x)$ iff $d^c_S(x) \in |\varphi|_S$. We equip S^* with the σ-algebra generated by the family of sets $|\varphi|_S$ for $\varphi :: S$. In this way, we define a *measurable space* S^*. Furthermore, notice that $|\varphi| \cap |\psi| = |\varphi \wedge \psi|$. So the family of generators is a π-system.

If we know the sort of φ is S, we sometimes drop the subscript and write $|\varphi|$ for $|\varphi|_S$. So $|\varphi|$ is the set of all theories of all points which satisfy φ.

Lemma 4.5. *If $f : X \to Y$ is a morphism of coalgebras $b : X \to TX$ and $c : Y \to TY$, then for every $S \in \mathsf{Ing}(T)$, $d^c_S \circ Sf = d^b_S$ for every $S \in \mathsf{Ing}(T)$. That is, coalgebra morphisms preserve description maps.*

Proof. By Lemma 4.1. $\square$

Lemma 4.6. *For all $c : X \to TX$, all $S \in \mathsf{Ing}(T)$:*

(1) *For all $\varphi : S$, $[\![\varphi]\!]^c_S = (d^c_S)^{-1}(|\varphi|)$.*
(2) *$d^c_S : SX \to S^*$ is measurable.*

Proof. The first point follows immediately from our definitions. For the second, recall that the σ-algebra on S^* is the one generated by the sets $|\varphi|$ for $\varphi :: S$. And for each such φ, its inverse image under d^c_S is the set $[\![\varphi]\!]^c_S$. This is a measurable subset of SX. $\square$

Example 4.7. We continue the development of Examples 3.2, 3.3, and 3.4. We take a look at the descriptions of the points in the coalgebra c from Example 3.4 by indicating a few representative formulas. $d^c_{Id}(x)$ contains true_{Id}, $[\mathsf{next}]\varphi_1$ (see Example 3.4), and also $[\mathsf{next}]\langle A, \mathsf{inr}\ \beta^p \mathsf{true}_{Id}\rangle$ for all A measurable subsets of $[0, 1]$ containing 1 and all $p \in [0, 1]$.

We present a table below with some sample theories (on the left) and some elements of those theories (on the right).

$d^c_{Id}(y), d^c_{Id}(z)$	true_{Id}
$d^c_{[0,1]}(1)$	All measurable $A \subseteq [0,1]$ containing 1
$d^c_U((q_1, q_2, q_3))$	$\beta^p(\mathsf{true}_{Id})$ for all $p \in [0,1]$
$d^c_V(\mathsf{inr}(1/3, 1/3, 1/3))$	$\mathsf{inr}\,\beta^1(\mathsf{true}_{Id})$
$d^c_V(\mathsf{inl}(1/2, 1/4, 1/4))$	$\mathsf{inl}\,\beta^1(\mathsf{true}_{Id})$
$d^c_V(\mathsf{inr}(0, 0, 1))$	$\mathsf{inr}\,\beta^1(\mathsf{true}_{Id})$
$d^c_T(\langle 1, \mathsf{inr}(1/3, 1/3, 1/3)\rangle)$	$\langle\{1\}, \mathsf{inr}\,\beta^1(\mathsf{true}_{Id})\rangle = \varphi_1$
	$\langle(1/2, 1], \mathsf{inr}\,\beta^1(\mathsf{true}_{Id})\rangle = \varphi_2$
$d^c_T(\langle 1/2, \mathsf{inl}(1/2, 1/4, 1/4)\rangle)$	$\langle(1/3, 2/3), \mathsf{inl}\,\beta^1(\mathsf{true}_{Id})\rangle = \psi_1$
	$\langle[1/2, 1), \mathsf{inl}\,\beta^1(\mathsf{true}_{Id})\rangle = \psi_2$
$d^c_T(\langle 0, \mathsf{inr}(0, 0, 1)\rangle)$	$\langle[0, 1], \mathsf{inr}\,\beta^1(\mathsf{true}_{Id})\rangle = \xi_1$
	$\langle[0, 1/100), \mathsf{inr}\,\beta^1(\mathsf{true}_{Id})\rangle = \xi_2$

Using the formulas above we also get:

$d^c_{Id}(x)$	$[\mathsf{next}]\varphi_1,\ [\mathsf{next}](\varphi_1 \wedge \varphi_2), [\mathsf{next}]\xi_1$
$d^c_{Id}(y)$	$[\mathsf{next}]\psi_1 \wedge [\mathsf{next}]\psi_2$
$d^c_{Id}(z)$	$[\mathsf{next}]\xi_1,\ [\mathsf{next}]\xi_2$
$d^c_U((1/3, 1/3, 1/3))$	$\beta^{1/3}[\mathsf{next}](\varphi_1 \wedge \varphi_2) = \varphi_3$
	$\beta^{1/3}[\mathsf{next}](\psi_1 \wedge \psi_2) = \varphi_4$
$d^c_U((1/2, 1/4, 1/4))$	$\beta^{1/2}[\mathsf{next}]\varphi_1 = \psi_3$
$d^c_U((0, 0, 1))$	$\beta^0[\mathsf{next}]\varphi_1 \wedge \beta^1[\mathsf{next}]\xi_1 = \xi_3$
$d^c_T(\langle 1, \mathsf{inr}(1/3, 1/3, 1/3)\rangle)$	$\langle\{1\}, \mathsf{inr}\,\varphi_3\rangle,\ \langle(3/5, 1], \mathsf{inr}\,\varphi_4\rangle$
$d^c_T(\langle 1/2, \mathsf{inl}(1/2, 1/4, 1/4)\rangle)$	$\langle(1/4, 3/4], \mathsf{inl}\,\psi_3\rangle$
$d^c_T(\langle 0, \mathsf{inr}(0, 0, 1)\rangle)$	$\langle[0, 4/5), \mathsf{inr}\,\xi_3\rangle$

Example 4.8. Following up on Example 3.6, here are some formulas from some of the descriptions:

$d^c_T(c(x))$	$\beta^{3/4}\mathsf{inl}[1/2, 1] = \varphi'_1,\ \beta^{1/4}\mathsf{inl}[0, 1/2) = \varphi'_2,\ \beta^0\mathsf{inr}\,\mathsf{true}_{Id}$
$d^c_T(c(y))$	$\beta^{1/2}\mathsf{inl}[1/2, 1] = \psi'_1,\ \beta^{1/2}\mathsf{inl}[0, 1/2) = \psi'_2,\ \beta^0\mathsf{inr}\,\mathsf{true}_{Id}$
$d^c_T(c(z))$	$\beta^{1/2}\mathsf{inl}[0, 1] \wedge \beta^{1/2}\mathsf{inr}\,\mathsf{true}_{Id} = \xi'_1$
$d^c_{Id}(x)$	$[\mathsf{next}]\varphi'_1 \wedge [\mathsf{next}]\varphi'_2$
$d^c_{Id}(y)$	$[\mathsf{next}]\psi'_1$
$d^c_{Id}(z)$	$[\mathsf{next}]\xi'_1$

4.2. Maps between canonical spaces

In the next lemmas, we build some measurable space mappings between spaces of the form S^*. The maps have additional properties stated in terms of the sets $|\varphi|$ which generate the σ-algebra structures. Most of the work in this section is very general (and straightforward). The only significant result is Lemma 4.11; that is where we use the measure theoretic results mentioned in Lemma 2.3 and Theorem 2.4.

Lemma 4.9. *Let $U \times V \in \mathsf{Ing}(T)$. There is a measurable map $\langle \pi_1, \pi_2 \rangle : (U \times V)^* \to U^* \times V^*$ such that*

(1) *For all coalgebras $c : X \to TX$, $\langle \pi_1, \pi_2 \rangle \circ d^c_{U \times V} = d^c_U \times d^c_V$:*

$$\begin{tikzcd}
(U \times V)X \arrow[d, "d^c_{U\times V}"'] \arrow[dr, "d^c_U \times d^c_V"] & \\
(U \times V)^* \arrow[r, "\langle \pi_1, \pi_2 \rangle"'] & U^* \times V^*
\end{tikzcd} \tag{4.3}$$

(2) *For $\varphi : U$ and $\psi : V$, $\langle \pi_1, \pi_2 \rangle^{-1}(|\varphi| \times |\psi|) = |\langle \varphi, \psi \rangle|$.*

Proof. First we define $\pi_1 : (U \times V)^* \to U^*$ by

$$\pi_1(s) = \{\varphi : U \mid \langle \varphi, \mathsf{true}_V \rangle \in s\}.$$

$\pi_2 : (U \times V)^* \to V^*$ is defined similarly. Then the map we want is the pair $\langle \pi_1, \pi_2 \rangle$.

We check that indeed $\pi_1(s) \in U^*$. Let $s \in (U \times V)^*$. Then there are X, c and $x \in (U \times V)(X)$ such that $s = d^c_{U\times V}(x)$. Then $x = \langle u, v \rangle \in (U \times V)X$. We claim that $\pi_1(s) = d^c_U(u)$. Indeed, for all $\varphi : U$, $\varphi \in \pi_1(s)$ iff $\langle \varphi, \mathsf{true} \rangle \in s = d^c_{U\times V}(\langle u, v \rangle)$ iff $\langle u, v \rangle \in [\![\langle \varphi, \mathsf{true}_V \rangle]\!]$ iff $u \in [\![\varphi]\!]$ iff $\varphi \in d^c_U(u)$. Similarly, $\pi_2(s) \in V^*$.

Let $s \in (U \times V)^*$. Let $\varphi : U$ and $\psi : V$. We have the following equivalences:

$$\begin{array}{lll}
 & s \in |\langle \varphi, \psi \rangle|_{U\times V} & \\
\text{iff} & \langle \varphi, \psi \rangle \in s & \text{by (4.2)} \\
\text{iff} & \langle \varphi, \mathsf{true}_V \rangle, \langle \mathsf{true}_U, \psi \rangle \ \in s & \\
\text{iff} & \varphi \in \pi_1(s) \ \& \ \psi \in \pi_2(s) & \\
\text{iff} & \pi_1(s) \in |\varphi|_U \ \& \ \pi_2(s) \in |\psi|_V & \text{by (4.2)} \\
\text{iff} & \langle \pi_1, \pi_2 \rangle(s) \in |\varphi|_U \times |\psi|_V. &
\end{array}$$

This completes the verification of part 2 . To see that $\langle \pi_1, \pi_2 \rangle$ is measurable, recall that one set of generators of the σ-algebra on $U^* \times V^*$ is the family of sets $|\varphi| \times |\psi|$, where $\varphi :: U$ and $\psi :: V$. The inverse image of this set is the measurable set $|\langle \varphi, \psi \rangle|$. $\square$

Lemma 4.10. *Let $U + V \in \mathsf{Ing}(T)$. There is a measurable map $\alpha : (U + V)^* \to U^* + V^*$ such that*

(1) *For all coalgebras* $c : X \to TX$, $\alpha \circ d^c_{U+V} = d^c_U + d^c_V$*:*
(2) *For* $\varphi : U$, $\alpha^{-1}(\mathsf{inl}_{U^*+V^*}(|\varphi|)) = |\mathsf{inl}\,\varphi|$*; and similarly for formulas of sort* V.

Proof. We define α by

$$\alpha(s) = \begin{cases} \mathsf{inl}_{U^*+V^*}(\{\varphi : U \mid \mathsf{inl}\,\varphi \in s\}), & \text{if } \mathsf{inl}\,\mathsf{true}_U \in s \\ \mathsf{inr}_{U^*+V^*}(\{\varphi : V \mid \mathsf{inr}\,\varphi \in s\}), & \text{if } \mathsf{inr}\,\mathsf{true}_V \in s \end{cases} \tag{4.4}$$

Let $s \in (U + V)^*$. Let $c : X \to TX$ be a coalgebra and let $x \in (U + V)(X)$ be such that $s = d^c_{U+V}(x)$. Then x is either $\mathsf{inl}\,u$ for some $u \in U(X)$, or $\mathsf{inr}\,v$ for some $v \in V(X)$; and both alternatives cannot simultaneously hold. In the first case, s contains $\mathsf{inl}\,\mathsf{true}_U$, and also the set $\{\varphi : U \mid \mathsf{inl}\,\varphi \in s\}$ is $d^c_U(u) \in U^*$. This is easy to verify: $\mathsf{inl}\,\varphi \in s = d^c_{U+V}(\mathsf{inl}\,u)$ iff $\mathsf{inl}\,u \in [\![\mathsf{inl}\,\varphi]\!] = \mathsf{inl}\,[\![\varphi]\!]$ iff $u \in [\![\varphi]\!]$. In the second case, s contains $\mathsf{inr}\,\mathsf{true}_V$, and also $\{\varphi : V \mid \mathsf{inr}\,\varphi \in s\} = d^c_V(v)$. This checks that indeed α maps to $U^* + V^*$.

In the notation from above, note that if $x = \mathsf{inl}\,u$, then $(\alpha \circ d^c_{U+V} \circ \mathsf{inl})(u) = (\alpha \circ d^c_{U+V})(x) = \mathsf{inl}\,d^c_U(u)$. The same is true for inr, and this leads easily to part *1* of this lemma. The measurability of α comes from the second statement in our lemma. Here is the verification: Let $s \in (U + V)^*$. Then

$$\begin{array}{lll} s \in |\mathsf{inl}\,\varphi| & \text{iff} & \mathsf{inl}\,\mathsf{true}_U \in s \text{ and } \mathsf{inl}\,\varphi \in s \\ & \text{iff} & \mathsf{inl}\,\mathsf{true}_U \in s \text{ and } \varphi \in \{\psi : U \mid \mathsf{inl}\,\psi \in s\} \\ & \text{iff} & \mathsf{inl}\,\mathsf{true}_U \in s \text{ and } \{\psi : U \mid \mathsf{inl}\,\psi \in s\} \in |\varphi| \\ & \text{iff} & \alpha(s) \in \mathsf{inl}_{U^*+V^*}(|\varphi|). \qquad \square \end{array}$$

Lemma 4.11. *Let* $\Delta S \in \mathsf{lng}(T)$. *There is a measurable map* $\epsilon : (\Delta S)^* \to \Delta(S^*)$ *such that*

(1) *For all coalgebras* $c : X \to TX$, $\epsilon \circ d^c_{\Delta S} = \Delta d^c_S$:
(2) *For* $\varphi :: S, \epsilon^{-1}(\beta^p(|\varphi|)) = |\beta^p\,\varphi|$.

Proof. We must go *from descriptions of measures to measures on descriptions.* Let $s \in (\Delta S)^*$. Then s is the set of formulas of sort ΔS satisfied by some element of some space $(\Delta S)X$. Recall that the formulas of sort ΔS are those of the form $\beta^p\varphi$ for $\varphi :: S$ together with $\mathsf{true}_{\Delta S}$ and conjunctions of these.

To define $\epsilon(s)$, let $c : X \to TX$ be a coalgebra, and let $\mu \in \Delta S(X)$ be such that $s = d^c_{\Delta S}(\mu)$. Let

$$\epsilon(s) = (\Delta d^c_S)\mu. \tag{4.5}$$

Then $\epsilon(s)$ depends on c, X, and μ, but no matter which are chosen, we have a probability measure on S^*. We must check that $\epsilon(s)$ is indeed independent of the choices of c, X, and μ, and also that ϵ so defined is measurable. However, for each $\varphi :: S$, we have

$$\begin{array}{lll} ((\Delta d^c_S)\mu)(|\varphi|) & = \mu((d^c_S)^{-1}(|\varphi|)) & \text{by the definition of } \Delta \\ & = \mu([\![\varphi]\!]^c_S) & \text{by Lemma 4.6} \\ & = \max\{p \mid \mu \in [\![\beta^p\varphi]\!]^c_{\Delta S}\} & \text{by equation (3.1)} \\ & = \max\{p \mid \beta^p\varphi \in s\} & \end{array}$$

The calculation above shows that the *number* $\epsilon(s)(|\varphi|)$ is independent of the choices of c, X, and μ: none of them appear in the expression $\max\{p : \beta^p\varphi \in s\}$. We still need to know that the *probability measure* $\epsilon(s)$ is independent of these choices; so far, we have only done that for sets of the form $|\varphi|$. However, the family sets of the form $|\varphi|$ is a π-system of generators of the σ-algebra on S^*, and by Theorem 2.4, there is at most one extension of any function defined on our family $\{|\varphi| \mid \varphi :: S\}$ to a probability measure on S^*. This means that $\epsilon(s)$ defined in (4.5) is indeed independent of our choices of c, X, and μ.

The independence tells us that for all c, X, and μ, $\epsilon(d^c_{\Delta S}(\mu)) = (\Delta d^c_S)\mu$. This for all μ shows that $\epsilon \circ d^c_{\Delta S} = \Delta d^c_S$, and we have part 1. For part 2 and hence for the measurability of ϵ, let $\beta^p\varphi :: \Delta S$. Then $\varphi :: S$, and

$$\begin{aligned}
\epsilon^{-1}(\beta^p|\varphi|) &= \{s \in (\Delta S)^* | \epsilon(s) \in \beta^p(|\varphi|)\} \\
&= \{s \in (\Delta S)^* | \epsilon(s)(|\varphi|) \geqslant p\} \\
&= \{s \in (\Delta S)^* | \beta^p\varphi \in s\} \qquad (*) \\
&= |\beta^p\varphi|.
\end{aligned}$$

Concerning the equivalence at the line marked (*), we recall the argument from above: if $\epsilon(s)(|\varphi|) \geqslant p$, then s contains $\beta^q\varphi$ for some $q \geqslant p$, namely for $q = \epsilon(s)(|\varphi|)$. But then s must also contain $\beta^p\varphi$, since all theories of all points have this monotonicity property. And conversely, if $\beta^p\varphi \in s$, then the largest q such that $\beta^q\varphi \in s$ is at least p.

By Lemma 2.3, the sets of the form $\beta^p|\varphi|$ generate the σ-algebra on $\Delta(S^*)$. So the equation $\epsilon^{-1}(\beta^p|\varphi|) = |\beta^p\varphi|$ proves the measurability of ϵ. $\square$

Lemma 4.12. *There is a measurable map* $[\mathsf{next}]^{-1} : Id^* \to T^*$ *such that*

(1) *For all coalgebras* $c : X \to TX$*, the diagram below commutes:*

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d^c_{Id}"'] & TX \arrow[d, "d^c_T"] \\
Id^* \arrow[r, "{[\mathsf{next}]^{-1}}"'] & T^*
\end{tikzcd} \tag{4.6}$$

(2) *For* $\varphi : T$*, the inverse image of* $|\varphi|_T$ *under* $[\mathsf{next}]^{-1}$ *is* $|[\mathsf{next}]\varphi|_{Id}$.

Proof. It should be noted that despite our notation, $[\mathsf{next}]^{-1}$ is not defined to be the inverse of a function "$[\mathsf{next}]$." We directly define $[\mathsf{next}]^{-1} : Id^* \to T^*$ by

$$[\mathsf{next}]^{-1}(s) = \{\varphi : T \mid [\mathsf{next}]\varphi \in s\}. \tag{4.7}$$

We check that this definition is proper and at the same time check part 1. Let $c : X \to TX$ and $x \in X$ be such that $s = d^c_{Id}(x)$. Then $c(x) \in TX$.

$$
\begin{aligned}
[\mathsf{next}]^{-1}(d^c_{Id}(x)) &= [\mathsf{next}]^{-1}(\{\varphi : Id \mid x \in [\![\varphi]\!]_{Id}\}) \\
&= \{\psi : T \mid [\mathsf{next}]\psi \in \{\varphi : Id \mid x \in [\![\varphi]\!]_{Id}\ \}\} \\
&= \{\psi : T \mid x \in [\![[\mathsf{next}]\psi]\!]_{Id}\ \} \\
&= \{\psi : T \mid c(x) \in [\![\psi]\!]_T\} \\
&= d^c_T(c(x)).
\end{aligned}
$$

At this point we have verified part 1. Here is the verification of part 2: For $s \in Id^*$, $s \in |[\mathsf{next}]\varphi|_{Id}$ iff $[\mathsf{next}]\varphi \in s$ iff $\varphi \in [\mathsf{next}]^{-1}(s)$ iff $[\mathsf{next}]^{-1}(s) \in |\varphi|_T$. This also leads to the measurability of $[\mathsf{next}]^{-1}$; recall that if $\varphi :: T$, then also $[\mathsf{next}]\varphi :: Id$.

This completes the proof. □

5. The spaces $S(Id^*)$

The final coalgebra theorem that we prove below builds a map $c^* : Id^* \to T(Id^*)$. So for this reason, we need to study the spaces $S(Id^*)$. The reader should not confuse these with the spaces S^* which we have already seen. These will re-appear shortly below.

At this point, we need to have a handle on the sets associated with formulas on the spaces $S(Id^*)$. We do this by defining for each $\varphi : S$ a subset $\widehat{\varphi} \subseteq S(Id^*)$.

For $\varphi : Id$, $\widehat{\varphi} = |\varphi|_{Id}$. For $A : M$, $\widehat{A} = A$. For $\langle\varphi,\psi\rangle : U \times V$, $\widehat{\langle\varphi,\psi\rangle} = \widehat{\varphi} \times \widehat{\psi}$. For $\varphi : U$, $\widehat{\mathsf{inl}\,\varphi} = \mathsf{inl}(\widehat{\varphi})$; and $\widehat{\mathsf{inr}\,\varphi} = \mathsf{inr}(\widehat{\varphi})$. For $\varphi :: S$, $\widehat{\beta^p\varphi} = \beta^p\widehat{\varphi}$. And for all S, $\widehat{\mathsf{true}}_S = S(Id^*)$, and $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cap \widehat{\psi}$.

Lemma 5.1. *There is a family of measurable maps $r_S : S^* \to S(Id^*)$ indexed by the ingredients of T such that the following hold:*

(1) *For all coalgebras $c : X \to TX$, the diagram below commutes:*

$$
\begin{tikzcd}
SX \arrow[d, "d^c_S"'] \arrow[dr, "Sd^c_{Id}"] & \\
S^* \arrow[r, "r_S"'] & S(Id^*)
\end{tikzcd}
\tag{5.1}
$$

(2) *For all $\varphi : S$, $r_S^{-1}(\widehat{\varphi}) = |\varphi|$.*

Proof. The maps $r_S : S^* \to S(Id^*)$ are defined by recursion on $\mathsf{Ing}(T)$.

For $S = Id$, we take r_S to be the identity on Id^*. Both parts of this lemma are immediate. It is easy to check part *2* by induction for conjunctions on all sorts. So we shall omit all reference to conjunction in the rest of this proof.

The constant functors M**.** We first note that each element $m \in M^*$ is a theory of some point in M. It is here that we use the fact that our language contains the singletons $\{x\}$. Hence for each such $m \in M^*$, there is a unique $x \in M$ such that $m = d^c_M(x)$, where c is an arbitrary coalgebra. We therefore get a bijection of M^* with $M = M(Id^*)$. We take $r_M : M^* \to M$ to be this bijection. Note that when $m = d^c_M(x)$, we have for all $A : M$ that $x \in A = \hat{A}$ iff $A \in m$. This leads quickly to part 2 of this lemma.

Probability measures. We define $r_{\Delta S}$ as $\Delta r_S \circ \epsilon$.

$$\begin{tikzcd}
\Delta SX \arrow[d, "d^c_{\Delta S}"'] \arrow[dr, "\Delta d^c_S"] \arrow[drr, bend left, "\Delta S d^c_{Id}"] & & \\
 & \Delta(S^*) \arrow[r, "\Delta r_S"'] & \Delta S(Id^*) \\
(\Delta S)^* \arrow[ur, "\epsilon"'] \arrow[urr, bend right, "r_{\Delta S}"'] & &
\end{tikzcd} \qquad (5.2)$$

The triangle on the left commutes by Lemma 4.11, and the one on the top by the induction hypothesis. Hence the outside of the diagram commutes, and we have part 1. For part 2:

$$\begin{aligned}
r^{-1}_{\Delta S}(\widehat{\beta^p\varphi}) &= \epsilon^{-1}(\Delta r_S)^{-1}\beta^p(\widehat{\varphi}) & \\
&= \epsilon^{-1}\beta^p(r_S)^{-1}(\widehat{\varphi}) & \text{by Lemma 2.1} \\
&= \epsilon^{-1}\beta^p(|\varphi|) & \text{by induction hypothesis} \\
&= |\beta^p\varphi| & \text{by Lemma 4.11}
\end{aligned}$$

Products. The argument here is almost the same. Given r_U and r_V with the desired properties, we define $r_{U\times V}$ to be $(r_U \times r_V) \circ \langle \pi_1, \pi_2 \rangle$. One verifies part *1* with a figure that is almost the same as that in (5.2) above. (The changes in the objects are that ΔS is now $U \times V$, and $\Delta(S^*)$ is $U^* \times V^*$; for the maps, the main change is that ϵ is now $\langle \pi_1, \pi_2 \rangle$.)

For part 2:

$$\begin{aligned}
r^{-1}_{U\times V}(\widehat{\langle\varphi,\psi\rangle}) &= \langle \pi_1, \pi_2 \rangle^{-1}((r_U \times r_V)^{-1}(\widehat{\varphi} \times \widehat{\psi})) & \\
&= \langle \pi_1, \pi_2 \rangle^{-1}(|\varphi| \times |\psi|) & \text{by induction hypothesis} \\
&= |\langle \varphi, \psi \rangle| & \text{by Lemma 4.9}
\end{aligned}$$

Coproducts. We take r_{U+V} to be $(r_U + r_V) \circ \alpha$. We use the diagram from (5.2), replacing $U \times V$ with $U + V$, and Lemma 4.11 with Lemma 4.10. The verification in part 2 is again basically the same as what we have seen. □

6. The final coalgebra for *T*

We now define the analog of the canonical model for our modal language $\mathcal{L}(T)$. Let $c^* : Id^* \to T(Id^*)$ be

$$r_T \circ [\mathsf{next}]^{-1} : \; Id^* \to T^* \to T(Id^*) \qquad (6.1)$$

We shall show that c^* is a final T-coalgebra. But first, here is a result inspired by the "Truth Lemma" of modal logic.

Lemma 6.1 (Truth Lemma)**.** *For all* $\varphi : S, [\![\varphi]\!]_S^{c^*} = \widehat{\varphi}$.

Proof. By induction on φ. The first base case of true_S uses $[\![\mathsf{true}]\!]_S^{c^*} = S(Id^*) = \widehat{\mathsf{true}}$.

The other base case is for $A : M$, where M is a constant functor in $\mathsf{Ing}(T)$. In this case, $[\![A]\!]_S^{c^*} = A = \widehat{A}$.

The inductive steps for all of the constructors besides $[\mathsf{next}]$ are easy. For example, the inductive step for $\beta^p\varphi : \Delta S$ is:

$$[\![\beta^p\varphi]\!]_{\Delta S}^{c^*} = \beta^p[\![\varphi]\!]_S^{c^*} = \beta^p\widehat{\varphi} = \widehat{\beta^p\varphi}.$$

So the crucial point comes with the inductive step for $[\mathsf{next}]\varphi$. In the following calculation, we use f as a notation for $[\mathsf{next}]^{-1}$ from Lemma 4.12.

We have

$$\begin{aligned}
[\![[\mathsf{next}]\varphi]\!]_{Id}^{c^*} &= (c^*)^{-1}([\![\varphi]\!]_T^{c^*}) \\
&= (c^*)^{-1}(\widehat{\varphi}) && \text{by induction hypothesis} \\
&= f^{-1}(r_T^{-1}(\widehat{\varphi})) && \text{by the definition of } c^* \text{ in (6.1)} \\
&= f^{-1}(|\varphi|_T) && \text{by Lemma 5.1} \\
&= |[\mathsf{next}]\varphi|_{Id} && \text{by Lemma 4.12} \\
&= \widehat{[\mathsf{next}]}\varphi
\end{aligned}$$

In the last line we used the fact that $[\mathsf{next}]\varphi$ is of sort Id; see the opening of Section 5. $\square$

Lemma 6.2. $d_{Id}^{c^*} = Id_{Id^*}$.

Proof. If $\varphi : Id$, then by the Truth Lemma, $[\![\varphi]\!]_{Id}^{c^*} = \widehat{\varphi} = |\varphi|$. So for $s \in Id^*$,

$$d_{Id}^{c^*}(s) = \{\varphi : Id \mid s \in [\![\varphi]\!]_{Id}^{c^*}\} = \{\varphi : Id \mid s \in |\varphi|\} = s. \qquad \square$$

Lemma 6.3. *For each coalgebra* $c : X \to TX, d_{Id}^c$ *is a morphism of coalgebras.*

Proof. Consider

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d_{Id}^c"'] & TX \arrow[d, "d_T^c"] \arrow[dr, "Td_{Id}^c"] & \\
Id^* \arrow[r, "{[\mathsf{next}]^{-1}}"'] & T^* \arrow[r, "r_T"'] & T(Id^*)
\end{tikzcd}$$

The square is Lemma 4.12, and the triangle is a special case of equation (5.1) in Lemma 5.1. $\square$

Theorem 6.4. $c^* : Id^* \to T(Id^*)$ *is a final coalgebra of T.*

Proof. Let $c : X \to TX$ be a T-coalgebra. By Lemma 6.3, d^c_{Id} is a coalgebra morphism. For the uniqueness, suppose that f is any morphism. Since f preserves descriptions, $d^{c^*}_{Id} \circ f = d^c_{Id}$. But by Lemma 6.2, $d^{c^*}_{Id} = Id_{Id^*}$. So $f = d^{c^*}_{Id} \circ f = d^c_{Id}$, just as desired. □

We conclude with an important corollary of our development. We know of no direct proof of Corollary 6.5 below.

Corollary 6.5. *For each $S \in \mathsf{Ing}(T)$, the map $r_S : S^* \to S(Id^*)$ is surjective.*

Proof. Consider the coalgebra $c^* : Id^* \to T(Id^*)$. By Lemma 5.1, $r_S \circ d^{c^*}_S = Sd^{c^*}_{Id}$. And by Lemma 6.2, $d^{c^*}_{Id} = Id_{Id^*}$. Thus $r_S \circ d^{c^*}_S = SId_{Id^*} = Id_{S(Id^*)}$. And this means that r_S is surjective. □

7. Variation: probabilistic Kripke polynomial functors on Set

In this section, we are interested in the category Set of sets rather than Meas. The first thing is to adapt the probability measure functor to Set. The most straightforward way to do this is to consider the *discrete probability measure functor* $\mathcal{D}$. A *discrete probability measure* on a set A is a function $\mu : A \to [0,1]$ such that

(1) μ has *finite support*: $\{a \in A \mid \mu(a) > 0\}$ is finite.
(2) $\sum_{a\in A} \mu(a) = 1$.

$\mathcal{D}(A)$ is the set of discrete probability measures on A. We make $\mathcal{D}$ into a functor by setting, for $f : A \to B$, $\mathcal{D}f(\mu)(b) = \mu(f^{-1}(b))$; this is $\sum\{\mu(a) : f(a) = b\}$. (As usual, we extend discrete probability measures from functions on A to functions on $\mathcal{P}(A)$ by summing, and we don't distinguish the original measure from its extension in our notation.)

Rößiger and then Jacobs considered the *Kripke polynomial functors* (KPF's) on Set. These are the functors on Set built from the identity functor, the finite power set functor, product and coproduct, fixed (finite or infinite) sets, and functions from a fixed set. The last of these works as follows: for each set E we have a functor $(\cdot)^E$. For each set a, a^E is the set of functions from E to a. And if $f : a \to b$, then $f^E : a^E \to b^E$ is given by $f^E(g) = f \circ g$. Adding the function space construct means that if S is a KPF and E is a set, then S^E is a KPF. We add to the constructs of the KPF's the discrete probability measure functor $\mathcal{D}$, and call the resulting class of functors Probabilistic Kripke Polynomial Functors (PKPFs). So if T is a PKPF, so is $\mathcal{D}T$. In this section, we check that the same general method of the foregoing part of our paper (with a few changes) also gives representations for final coalgebras for PKPFs.

To avoid double subscripts or confusion with our notation $\mathcal{P}$ for the power set functor, we shall use $\mathcal{Q}$ for the *finite power set* functor on Set. Being a functor, we shall apply $\mathcal{Q}$ to functions as well as sets, writing, e.g., $\mathcal{Q}r(X)$ for the image $r[X]$ of the finite set X under r.

It is well-known in the coalgebra literature that all KPFs have final coalgebras. Most of the proofs extend to the class of PKPFs as well. One can prove this by checking that all such functors are bounded and then using the much more general fact that bounded functors on Set have final coalgebras (more generally, one may use the theory of accessible functors). Another way to get the result is to use Aczel's Special Final Coalgebra Theorem from his book [21], or one of the descendants of this result. Alternatively, one can use a logical approach, as done in papers such as [3,22,1,2]. This is the approach that we take. However, our work is a bit different than in the cited works since our final coalgebra is based on the satisfied theories rather than the maximal consistent ones in some logical system. This means that our result is actually weaker, since we do not obtain a completeness result. On the other hand, we believe that it is is easier to get the final coalgebra this way. And the general method works even in the absence of a logic, as we have seen in the work on measurable spaces.

Operations on sets. Let T be a PKPF. The syntax and semantics of $\mathcal{L}(T)$ are based on operations on sets associated to the functors S^E, $\mathcal{Q}$, and $\mathcal{D}$. An operation on sets is a set-indexed family of maps. For example, as we shall see shortly, each $e \in E$ gives an operation (e). This technically is a family of maps $(e)_a : \mathcal{P}(a) \to \mathcal{P}(a^E)$. Our plan is to isolate a general naturality condition on such operations in Lemma 7.1. Then we take each of our operations as a syntactic symbol in $\mathcal{L}(T)$. The semantics of $\mathcal{L}(T)$ is strongly based on the particular operations we define below. The properties of them in Lemma 7.1 turn out to abstract the basic properties of the semantics. The more subtle properties, the ones which are the keys to the finality result, are explored in Section 7.2 below.

We next turn to the operations themselves. Let $w \subseteq a$.

For the function set functor S^E, each $e \in E$ gives an operation (e) defined by

$$(e)_a(w) = \{\tau \in a^E \mid \tau(e) \in w\}.$$

For the finite power set functor $\mathcal{Q}$, we have the operation $\Box$ given by

$$\Box_a(w) = \mathcal{Q}(w) = \{y \mid y \text{ is finite and } y \subseteq w\}.$$

(So the subscripts here do not do any work; we shall drop them shortly.) For the discrete probability functor $\mathcal{D}$, each $p \in [0,1]$ gives an operation β^p by

$$\beta^p_a(w) = \{\mu \in \mathcal{D}(a) \mid \mu(w) \geqslant p\}.$$

Lemma 7.1. *For all $g : a \to b$, $w \subseteq b$, $e \in E$, and $p \in [0,1]$,*

(1) $(e)_a(g^{-1}(w)) = (g^E)^{-1}((e)_b(w))$.
(2) $\Box_a(g^{-1}(w)) = (\mathcal{Q}g)^{-1}\Box_b w$.
(3) $\beta^p_a(g^{-1}(w)) = (\mathcal{D}g)^{-1}\beta^p_b w$.

Proof.

$$\begin{aligned}(1)\quad (e)_a(g^{-1}(w)) &= \{\tau \in a^E \mid \tau(e) \in g^{-1}(w)\}\\ &= \{\tau \in a^E \mid (g \circ \tau)(e) \in w\}\\ &= (g^E)^{-1}(\{\tau' \in b^E \mid \tau'(e) \in w\})\\ &= (g^E)^{-1}((e)_b(w))\end{aligned}$$

$$
\begin{aligned}
(2)\quad \Box_a(g^{-1}(w)) &= \{s \in \mathcal{Q}a \mid s \subseteq g^{-1}(w)\}\\
&= \{s \in \mathcal{Q}a \mid g[s] \subseteq w\}\\
&= \{s \in \mathcal{Q}a \mid (\mathcal{Q}g)s \subseteq w\}\\
&= \{s \in \mathcal{Q}a \mid (\mathcal{Q}g)s \in \mathcal{Q}w\}\\
&= (\mathcal{Q}g)^{-1}\Box_b w
\end{aligned}
$$

(3) $\beta^p_a(g^{-1}(w)) = \{\mu \in \mathcal{D}(a) | \mu(g^{-1}(w)) \geqslant p\} = \{\mu \in \mathcal{D}(a) | ((\mathcal{D}g)\mu)(w) \geqslant p\} = (\mathcal{D}g)^{-1}\beta^p_b w.$ $\Box$

7.1. Syntax and semantics

Let T be a PKPF. We define the *ingredients of* T in the obvious way: $\mathsf{Ing}(Id) = \{Id\}$; $\mathsf{Ing}(A) = \{A, Id\}$; $\mathsf{Ing}(U \times V) = \{U \times V\} \cup \mathsf{Ing}(U) \cup \mathsf{Ing}(V)$, and similarly for $U + V$; $\mathsf{Ing}(S^E) = \{S^E\} \cup \mathsf{Ing}(S)$; $\mathsf{Ing}(\mathcal{Q}S) = \{\mathcal{Q}S\} \cup \mathsf{Ing}(S)$; and $\mathsf{Ing}(\mathcal{D}S) = \{\mathcal{D}S\} \cup \mathsf{Ing}(S)$.

We construct a language $\mathcal{L}(T)$ as follows. We take $\mathsf{true}_S : S$ for each ingredient S, and we have conjunction and negation on each S as well. (We believe that it is necessary to have negation for the results of this section.) If for some set A, the associated constant functor $A \in \mathsf{Ing}(T)$, then each *element* $a \in A$ is a formula of sort A. (This is basically the same thing as taking the singletons from A.) Further, if $S^E \in \mathsf{Ing}(T)$ for some set E, $\varphi : S$ and $e \in E$, then $(e)\varphi : S^E$. If $\mathcal{Q}S \in \mathsf{Ing}(T)$ and $\varphi : S$, then $\Box\varphi : \mathcal{Q}S$. (Note that this operator does not come with subscripts.) Finally, in a similar fashion to what we had for the measurable spaces, if $\mathcal{D}S \in \mathsf{Ing}(T)$, then for each real number $p \in [0,1]$ and $\varphi : S$ we have $\beta^p\varphi : \mathcal{D}S$.

An important property of the syntax is that for all sorts S, the set $\mathcal{L}_S$ of formulas of sort S is the closure under the boolean operations of true, $\wedge$, and $\neg$ of a set of formulas that is easily specified in terms of $\mathcal{L}_{S'}$ for some other ingredient S' (or perhaps two ingredients, in the case when S is a product or coproduct).

The semantics of the language is given in Fig. 3.

General discussion of the changes. Most of the rest of the results from earlier in the paper go through with only minor changes, dropping the word "measurable" and anything having to do with the measurable space structure. The main differences are in Lemma 5.1 and the Truth Lemma 6.1.

Here is how the results from earlier in the paper adapt to the category of sets. Lemma 4.1 is the same, except that we add induction steps for the formulas $(e)\varphi$ and for the formulas $\Box\varphi$. In all cases the property from Lemma 7.1 does the work. We also need to add an easy step for negation. Lemma

$$
\begin{array}{rclrcl}
[\![\mathsf{true}]\!]^c_S & = & SX & [\![\langle\varphi,\psi\rangle]\!]^c_{U\times V} & = & [\![\varphi]\!]^c_U \times [\![\psi]\!]^c_V\\
[\![\mathsf{inl}\,\varphi]\!]^c_{U+V} & = & \mathsf{inl}([\![\varphi]\!]^c_U) & [\![a]\!]^c_A & = & \{a\}\\
[\![\mathsf{inr}\,\varphi]\!]^c_{U+V} & = & \mathsf{inr}([\![\varphi]\!]^c_V) & [\![(e)\varphi]\!]^c_{S^E} & = & (e)[\![\varphi]\!]^c_S\\
[\![\varphi\wedge\psi]\!]^c_S & = & [\![\varphi]\!]^c_S \cap [\![\psi]\!]^c_S & [\![\Box\varphi]\!]^c_{\mathcal{Q}S} & = & \mathcal{Q}([\![\varphi]\!]^c_S)\\
[\![\neg\varphi]\!]^c_S & = & SX \setminus [\![\varphi]\!]^c_S & [\![\beta^p\varphi]\!]^c_{\mathcal{D}S} & = & \beta^p([\![\varphi]\!]^c_S)\\
 & & & [\![[\mathsf{next}]\varphi]\!]^c_{Id} & = & c^{-1}([\![\varphi]\!]^c_T)
\end{array}
$$

Fig. 3. The semantics of $\mathcal{L}(T)$ in our work on Set.

4.5 is the same. Part 1 of Lemma 4.6 is the same, and part 2 does not apply. For $\varphi : S$, we define $\widehat{\varphi}$ as before, adding these clauses: $\widehat{\neg\varphi} = S(Id^*) \setminus \widehat{\varphi}$, $\widehat{(e)\varphi} = (e)(\widehat{\varphi})$, and $\widehat{\Box\varphi} = \Box(\widehat{\varphi})$.

We shall discuss Lemma 5.1 below, since it is where all the changes happen. The Truth Lemma 6.1 requires a trivial induction step for negation. None of the rest of the results in Section 6 mention measures, so they go through automatically.

7.2. Maps between canonical spaces

As in Section 4.2, we shall need some maps between various of our spaces of the form S^*. Obviously, measurability plays no role in the results of this section.

Lemma 7.2. *Let $S^E \in \mathsf{lng}(T)$. There is a function $\eta : (S^E)^* \to (S^*)^E$ such that*

(1) *For all coalgebras $c : X \to TX$, $\eta \circ d^c_{S^E} = (d^c_S)^E$.*
(2) *For $\varphi : S, \eta^{-1}((e)|\varphi|) = |(e)\,\varphi|$.*

Proof. Let $f \in S^E(X) = (SX)^E$. We define η by

$$\eta(d^c_{S^E}(f)) = d^c_S \circ f. \tag{7.1}$$

We must make sure that this is well-defined. This boils down to the following assertion:

$$d^c_{S^E}(f) = d^{c'}_{S^E}(f') \quad \text{iff} \quad \text{for all } e \in E, d^c_S(f(e)) = d^{c'}_S(f'(e)).$$

In both directions, we use induction on the overall language. Going left-to-right, note that for $\varphi : S$ we have for all $e \in E$ that $\varphi \in d^c_S(f(e))$ iff $(e)\varphi \in d^c_{S^E}(f)$ iff $(e)\varphi \in d^{c'}_{S^E}(f')$ iff $\varphi \in d^{c'}_S(f'(e))$. The rest of the induction steps are for the boolean connectives of sort S; these steps are easy. The right-to-left direction is similar. The main point is that the formulas of sort S^E are the boolean closure of the formulas $(e)\varphi$ for $\varphi : S$.

Then

$$\begin{aligned}(\eta \circ d^c_{S^E})(f) &= d^c_S \circ f\\ &= (d^c_S)^E(f)\end{aligned}$$

This completes part 1 of this lemma.

For part 2,

$$\begin{aligned}\eta^{-1}((e)|\varphi|) &= \{s \in (S^E)^* \mid \eta(s) \in (e)|\varphi|\}\\ &= \{s \in (S^E)^* \mid \eta(s)(e) \in |\varphi|\}\\ &= \{s \in (S^E)^* \mid (e)\varphi \in s\} \qquad \text{see below}\\ &= |(e)\varphi|\end{aligned}$$

Here is our justification for the marked line. Let f be such that $d^c_{S^E}(f) = s$. Then $\eta(s)(e) = d^c_S(f(e))$. Then $(e)\varphi \in s = d^c_{S^E}(f)$ iff $\varphi \in d^c_S(f(e))$ iff $d^c_S(f(e)) \in |\varphi|$ iff $\eta(s)(e) \in |\varphi|$. $\square$

Lemma 7.3. *Let $\mathcal{Q}S \in \mathsf{lng}(T)$. There is a function $\zeta : (\mathcal{Q}S)^* \to \mathcal{Q}(S^*)$ such that*

(1) *For all coalgebras $c : X \to TX$, $\zeta \circ d^c_{\mathcal{Q}S} = \mathcal{Q}d^c_S$.*

(2) *For* $\varphi : S, \zeta^{-1}(\mathcal{Q}(|\varphi|)) = |\Box\,\varphi|$.

Proof. Let $s \in (\mathcal{Q}S)^*$. We set

$$\zeta(s) = \bigcap_{s \in |\Box\varphi|} |\varphi|. \tag{7.2}$$

We must check that the intersection above is a *finite* subset of S^* and that for all coalgebras $c : X \to TX$, $\zeta \circ d^c_{\mathcal{Q}S} = \mathcal{Q}d^c_S$. For all of this, let $Y \in \mathcal{Q}S(X)$ be such that $s = d^c_{\mathcal{Q}S}(Y)$. Y is finite, so list it as $\{y_1, \ldots, y_n\}$. We claim that

$$\{d^c_S(y_1), \ldots, d^c_S(y_n)\} = \bigcap_{s \in |\Box\varphi|} |\varphi|. \tag{7.3}$$

To see this, let $y_i \in Y$. Let φ be such that $s \in |\Box\varphi|$. Then $Y \in [\![\Box\varphi]\!]_{\mathcal{Q}S}$. So $y_i \in [\![\varphi]\!]_S$. Thus $d^c_S(y_i) \in |\varphi|$. Since φ is arbitrary, $d^c_S(y_i)$ belongs to the right side of (7.3). In the other direction, suppose that $z \in S^*$ is such that z belongs to $|\varphi|$ whenever $s \in |\Box\varphi|$. We claim that $z = d^c_S(y_i)$ for some i. For if not, then for each i there is some $\psi_i : S$ such that $\psi_i \in z$ and $\neg\psi_i \in d^c_S(y_i)$. Then $Y \in [\![\Box(\neg\psi_1 \vee \cdots \vee \neg\psi_n)]\!]_{\mathcal{Q}S}$. Since $s = d^c_{\mathcal{Q}S}(Y)$, we see that s contains $\Box(\neg\psi_1 \vee \cdots \vee \neg\psi_n)$. But then by the definition of z, $z \in |\neg\psi_1 \vee \cdots \vee \neg\psi_n|$. This contradicts the fact that $\psi_i \in z$ for all i.

So at this point, we know (7.3). This verifies that the definition of $\zeta(s)$ in (7.2) is a finite subset of S^*. It also follows from (7.2) and (7.3) that for the coalgebra c and the element Y in our discussion above, $\zeta(d^c_{\mathcal{Q}S}(Y)) = \mathcal{Q}(d^c_S(Y))$. But the definition of the function ζ was independent of the coalgebra that we used to study it. And so we conclude that for all coalgebras c, $\zeta \circ d^c_{\mathcal{Q}S} = \mathcal{Q}d^c_S$.

We next consider part 7.5 of this lemma. The proof is by induction on sentences of sort $\mathcal{Q}S$. The main work is for sentences of the form $\Box\varphi$, with $\varphi : S$. We calculate:

$$\begin{aligned}\zeta^{-1}\mathcal{Q}(|\varphi|) &= \{s \in (\mathcal{Q}S)^* \mid \zeta(s) \subseteq |\varphi|\} \\ &= \{s \in (\mathcal{Q}S)^* \mid \Box\varphi \in s\} \qquad \text{see below} \\ &= |\Box\varphi|.\end{aligned}$$

Here is the argument for the indicated point. By (7.2), if $\Box\varphi \in s$, then $\zeta(s) \subseteq |\varphi|$. The other case is when $\neg\Box\varphi \in s$. Let $c : X \to TX$ be a coalgebra, and let $Y = \{y_1, \ldots, y_n\} \in \mathcal{Q}S(X)$ be such that $s = d^c_{\mathcal{Q}S}(Y)$. Let i be such that $\neg\varphi \in d^c_S(y_i)$. That is, $d^c_S(y_i) \notin |\varphi|$. By Eq (7.3) above, $\zeta(s) = \{d^c_S(y_j) \mid j \leqslant n\}$. At least one element of this set, $d^c_S(y_i)$, does not belong to $|\varphi|$. So $\zeta(s) \not\subseteq |\varphi|$. $\Box$

Lemma 7.4. *Let* $\mathcal{D}S \in \mathsf{Ing}(T)$. *There is a function* $\theta : (\mathcal{D}S)^* \to \mathcal{D}(S^*)$ *such that*

(1) *For all coalgebras* $c : X \to TX, \theta \circ d^c_{\mathcal{D}S} = \mathcal{D}d^c_S$.
(2) *For* $\varphi : S, \theta^{-1}(\beta^p(|\varphi|)) = |\beta^p\,\varphi|$.

Proof. Let $s \in (\mathcal{D}S)^*$. For a moment, fix X, c, and $\mu \in \mathcal{D}SX$ such that $s = d^c_{\mathcal{D}S}(\mu)$. We define

$$\theta(s) = (\mathcal{D}d^c_S)(\mu). \tag{7.4}$$

We claim that for each $s \in (\mathcal{D}S)^*$ and $y \in S^*$,

$$\theta(s)(y) = \max\{p \mid \text{for all } \varphi \in y, \beta^p\varphi \in s\}. \tag{7.5}$$

Let q be the value on the right side of (7.5), and let $q' = (\mathcal{D}d_S^c)\mu(y) = \mu((d_S^c)^{-1}y)$ be the value on the left. Let $w = (d_S^c)^{-1}(y) \cap Supp(\mu)$. Note that $q' = \mu(w)$. For any $\varphi \in y$ we have $t \in [\![\varphi]\!]_S^c$ for all $t \in w$. So $\mu([\![\varphi]\!]_S^c) \geqslant \mu(w) = q'$. This implies that $\mu \in [\![\beta^{q'}\varphi]\!]_{\mathcal{D}S}^c$, that is, $\beta^{q'}\varphi \in d_{\mathcal{D}S}^c(\mu) = s$. Thus $q = \max\{p \mid \forall\varphi \in y, \beta^p\varphi \in s\} \geqslant q'$.

Recall that $Supp(\mu)$ is a finite set. Let ψ be a formula which holds of all elements of w but of no elements of $Supp(\mu) \setminus w$. (In more detail: if $w = \emptyset$, then let $\psi = \neg\mathsf{true}$. If $Supp(\mu) \subseteq w$, then let $\psi = \mathsf{true}$. Otherwise, let $t \in w$. For each $u \notin w$, t and u have different descriptions. So for each $u \in Supp(\mu) \setminus w$ we may find ψ_u such that $t \in [\![\psi_u]\!]_S^c$ and $u \in [\![\neg\psi_u]\!]_S^c$. Let ψ be the conjunction of the finite set of ψ_u for $u \in Supp(\mu) \setminus w$. Note that all points in w have the same description, so they all satisfy ψ.) Now $[\![\psi]\!]_S^c$ is the disjoint union of w and $[\![\psi]\!]_S^c \setminus Supp(\mu)$. The second set here has μ-measure 0. Using (3.1) we get:

$$\max\{p \mid \beta^p\psi \in s\} = \max\{p \mid \mu \in [\![\beta^p\psi]\!]_{\mathcal{D}S}^c\} = \mu([\![\psi]\!]_S^c) = \mu(w).$$

It follows that

$$q' = \mu(w) = \max\{p \mid \beta^p\psi \in s\} \geqslant q.$$

So at this point we know that equation (7.5) holds. This means that the definition in (7.4) is independent of the choice of X, c, and μ. Turning things around, we see that for all coalgebras $c : X \to TX$, and all $\mu \in \mathcal{D}SX$, $(\theta \circ d_{\mathcal{D}S}^c)(\mu) = \mathcal{D}d_S^c(\mu)$. Since μ is arbitrary, point 1 in our lemma holds.

To establish point *2* , we first need to prove the following fact:

For all $s \in (\mathcal{D}S)^*$ and $\varphi : S$, $\theta(s)|\varphi| \geqslant p$ if and only if $\beta^p\varphi \in s$. For this,

$$\begin{aligned}
\theta(s)|\varphi| &= \textstyle\sum_{y\in|\varphi|} \theta(x)(y) \\
&= \textstyle\sum_{\varphi\in y} \mu((d_S^c)^{-1}(y)) \\
&= \mu(\cup_{\varphi\in y}(d_S^c)^{-1}(y)) && \text{the sets } (d_S^c)^{-1}(y) \text{ are disjoint} \\
& && \text{for different values of } y \\
&= \mu([\![\varphi]\!]_S^c).
\end{aligned}$$

The last line is explained by the following equivalences: $t \in \bigcup_{\varphi\in y}(d_S^c)^{-1}(y)$ iff there exists y such that $\varphi \in y$ and $t \in (d_S^c)^{-1}(y)$; this holds iff $\varphi \in d_S^c(T)$ iff $t \in [\![\varphi]\!]_S^c$.

Thus $\theta(s)|\varphi| \geqslant p$ iff $\mu([\![\varphi]\!]_S^c) \geqslant p$ iff $\mu \in [\![\beta^p\varphi]\!]_{\mathcal{D}S}^c$ iff $\beta^p\varphi \in d_{\mathcal{D}S}^c(\mu) = s$.

We now check point 2 of this lemma:

$$\begin{aligned}
\theta^{-1}(\beta^p|\varphi|) &= \{s \in (\mathcal{D}S)^* \mid \theta(s) \in \beta^p|\varphi|\} \\
&= \{s \in (\mathcal{D}S)^* \mid \theta(s)|\varphi| \geqslant p\} \\
&= \{s \in (\mathcal{D}S)^* \mid \beta^p\varphi \in s\} && \text{by the observation above} \\
&= |\beta^p\varphi|
\end{aligned}$$

This completes the proof. $\square$

Lemma 7.5. *There is a family of maps $r_S : S^* \to S(Id^*)$ indexed by the ingredients of T such that the following hold:*

(1) *For all coalgebras $c : X \to TX$, $r_S \circ d^c_S = Sd^c_{Id}$.*
(2) *For all $\varphi : S$, $r_S^{-1}(\widehat{\varphi}) = |\varphi|$.*

Proof. By induction on ingredients S of T. The base cases and the induction steps for products, and coproducts are the same as in Lemma 5.1.

We treat in detail the induction step for functors $\mathcal{Q}S$. Let $r_{\mathcal{Q}S}$ be $\mathcal{Q}r_S \circ \zeta$. Then for part 1 we have that

$$r_{\mathcal{Q}S} \circ d^c_{\mathcal{Q}S} = \mathcal{Q}r_S \circ \zeta \circ d^c_{\mathcal{Q}S} = \mathcal{Q}r_S \circ \mathcal{Q}d^c_S = \mathcal{Q}(r_S \circ d^c_S) = \mathcal{Q}Sd^c_{Id}.$$

For part *2* we have:

$$\begin{aligned} r^{-1}_{\mathcal{Q}S}(\widehat{\Box\varphi}) &= r^{-1}_{\mathcal{Q}S}(\mathcal{Q}(\widehat{\varphi})) \\ &= \zeta^{-1}(\mathcal{Q}r_S)^{-1}(\mathcal{Q}\widehat{\varphi}) && \text{by def. of } r_{\mathcal{Q}S} \\ &= \zeta^{-1}(\mathcal{Q}(r_S^{-1}\widehat{\varphi})) && \text{by Lemma 7.1} \\ &= \zeta^{-1}(\mathcal{Q}(|\varphi|)) && \text{by induction hypothesis} \\ &= |\Box\varphi| && \text{by Lemma 7.3} \end{aligned}$$

Technically, part 2 is an induction on formulas of sort $\mathcal{Q}S$. The steps above constitute the base case of the induction. The steps for negation and conjunction are immediate from the observation that inverse images of functions preserve complements and intersections.

In a similar manner, we define r_{S^E} by $r_{S^E} = r^E_S \circ \eta$ and $r_{\mathcal{D}S}$ by $r_{\mathcal{D}S} = (\mathcal{D}r_S) \circ \theta$. □

8. Applications and future work

In writing on this topic, we faced the choice of whether to present the logical systems in concrete or abstract forms. We chose to be concrete; we can only hope that more readers would approve rather than disapprove of our choice. This means that we did not make all of the connections between our work and existing work in coalgebraic modal logic. The main notion which we left out was that of a *predicate lifting*. This is an abstraction of what we called an *operation on sets* in Section 7. The connection between predicate liftings and coalgebraic modal logic was first indicated by Jacobs [3] and studied by him and others in several papers. We know that our logical systems may also be defined in terms of predicate liftings. Moving to a more abstract setting might illuminate things, and it might facilitate other work on logical representations of final coalgebras. (At the same time, moving to a more abstract setting should not make any of the measure-theoretic arguments easier.)

We conclude with a short discussion of some applications of the existence of final coalgebras on Meas. First, consider the *transition probability functions* studied in Desharnais et al. [9]. These are essentially the coalgebras of Δ. In addition, the *labelled Markov processes* in the same paper are also essentially coalgebras of the subprobability measure functor $S(X) = \Delta(X + 1)$. Our main result, Theorem 6.4, implies that there are final coalgebras for all of these functors. More importantly, when someone next proposes a new extension of these notions, our results will already imply the existence of the final coalgebra for the new notion (provided, of course, that it may be expressed as a measure polynomial functor).

We might also note that in our work, the restriction to *analytic spaces* is not needed. To be sure, if a measure polynomial functor T has the property that all constant ingredients are analytic spaces, then the final coalgebra of T will again be analytic. This is a corollary to the proof of our result; under our hypotheses, it is easy to check that the spaces S^* are countably generated (use $\beta^p\varphi$ for *rational* p) and separate the points.

Further, one may take a look at the functors considered in [23]. The authors there consider some functors in the class of functors on Set containing the identity, constant functors, and closed under products, coproducts, powerset, $\mathcal{D}_\omega$ and constant exponents. $\mathcal{D}_\omega X$ represents here the set of all probability distributions with finite or countable support.

If one replaces the power set functor with the finite power set functor $\mathcal{Q}$, and also $\mathcal{D}_\omega$ with the functor $\mathcal{D}$ of discrete probability measures, then the resulting class is what we called the PKPF functors. We showed that every PKPF functor T has a final coalgebra, and we have a representation of the final coalgebra in terms of satisfied theories in $\mathcal{L}(T)$. If one wants to work with the countable power set functor and to $\mathcal{D}_\omega$, it would be necessary to add countable conjunctions to $\mathcal{L}(T)$.

In the first part of the paper, we considered the category Meas. Hence we could not include the power set functor, or any measure-theoretic analog of S^E. It's worth noting that there is no natural way of taking exponents in Meas (see [24]), so this explains our omission. Of course, if the exponent were finite, one can get the desired coalgebras using products, and a bit more of work along these lines should extend the results to (countable) infinite products of measurable spaces.

Theorem 6.4 yields final coalgebras for the functors which build deterministic automata $((Id + 1)^A)$, Markov chains (Δ), reactive systems $((\Delta + 1)^A)$, generative systems $(\Delta(A \times Id) + 1)$, and stratified systems $(\Delta + (A \times Id) + 1)$. A here denotes a finite set.

There are still open questions, some of which are under investigation. Are the languages $\mathcal{L}(T)$ presented here are strong enough to characterize bisimulations for all measure polynomial functors T, despite the fact that these might not preserve weak pullbacks? We would be interested in other finality results that can be proved using either the method of this paper or that of Viglizzo [12]. Concerning the logical languages used in this paper, are there logical systems for $\mathcal{L}(T)$ (or extensions with negation) that capture the natural notions of validity?

Acknowledgments

We are grateful to two anonymous referees and also to Aviad Heifetz and Martin Meier for insightful comments, criticism, and questions.

References

[1] M. Rößiger, Languages for coalgebras on datafunctors, in: CMCS'99 Coalgebraic Methods in Computer Science (Amsterdam, 1999), Electronic Notes Theoretical Computer Science, vol. 19, Elsevier, Amsterdam, 1999, pp. 22 (electronic).

[2] M. Rößiger, From modal logic to terminal coalgebras, Theor. Comput. Sci. 260 (1-2) (2001) 209–228, Coalgebraic methods in computer science (Lisbon, 1998).
[3] B. Jacobs, Many-sorted coalgebraic modal logic: a model-theoretic study, Theor. Inform. Appl. 35 (1) (2001) 31–59, Coalgebraic methods in computer science (Berlin, 2000).
[4] C. Kupke, A. Kurz, Y. Venema, Stone coalgebras, in: H.P. Gumm (Ed.), Electronic Notes in Theoretical Computer Science, vol. 82, Elsevier, Amsterdam, 2003.
[5] J.C. Harsanyi, Games with incomplete information played by "Bayesian" players. I. The basic model, Manage. Sci. 14 (1967) 159–182.
[6] W. Böge, T. Eisele, On solutions of Bayesian games, Internat. J. Game Theory 8 (4) (1979) 193–215.
[7] A. Heifetz, D. Samet, Topology-free typology of beliefs, J. Econ. Theory 82 (2) (1998) 324–341.
[8] L.S. Moss, I.D. Viglizzo, Harsanyi type spaces and final coalgebras constructed from satisfied theories, Electronic Notes in Theoretical Computer Science 106 (2004) 279–295.
[9] J. Desharnais, A. Edalat, P. Panangaden, Bisimulation for labelled Markov processes, Inform. Comput. 179 (2) (2002) 163–193, lICS'97 (Warsaw).
[10] M. Mislove, J. Ouaknine, D. Pavlovic, J. Worrell, Duality for labelled Markov processes, in: I. Walukiewicz (Ed.), Proceedings of FoSSaCS, Lecture Notes in Computer Sciences, vol. 2987, Springer, Berlin, 2004, pp. 393–407.
[11] M. Giry, A categorical approach to probability theory, in: Categorical aspects of topology and analysis, Springer Lecture Notes in Mathematics, vol. 915, Springer, Berlin-New York, 1982, pp. 68–85.
[12] I.D. Viglizzo, Final sequences and final coalgebras for measurable spaces, in: José Luiz Fiadeiro et al. (Eds.), Algebra and Coalgebra in Computer Science: First International Conference, CALCO 2005, Swansea, UK, September 3–6, 2005. Proceedings, vol. 3629, Springer-Verlag GmbH, 2005, pp. 395–407.
[13] I.D. Viglizzo. Coalgebras on measurable spaces. Ph.D. thesis, Indiana University (July 2005).
[14] P. Billingsley, Probability and Measure, 2nd Edition, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, New York, 1986.
[15] E.S. Andersen, B. Jessen, On the introduction of measures in infinite product sets, Danske Vid. Selsk. Mat-Fys. Medd. 25 (1948) 8.
[16] P.R. Halmos, Measure Theory, D, Van Nostrand Company, NewYork, NY, 1950, xi + 304 pp.
[17] J. Worrell, Terminal sequences for accessible endofunctors, in: CMCS'99 Coalgebraic Methods in Computer Science (Amsterdam, 1999), Electronic Notes in Theoretical Computer Science, vol. 19, Elsevier, Amsterdam, 1999, pp. 15 (electronic).
[18] J.F. Mertens, S. Zamir, Formulation of Bayesian analysis for games of incomplete information, Internat. J. Game Theory 14 (1984) 1–29.
[19] A. Heifetz, The Bayesian formulation of incomplete information—the non-compact case, Internat. J. Game Theory 21 (1993) 329–338.
[20] F. van Breugel, M. Mislove, J. Ouaknine, J. Worrell, Domain theory, testing and simulation for labelled Markov processes, Theor. Comput. Sci. 333 (1-2) (2005) 171–197.
[21] P. Aczel, Non-Well-Founded Sets, CSLI Lecture Notes, vol. 14, CSLI Publications, Stanford, 1988.
[22] A. Kurz, Specifying coalgebras with modal logic, Theor. Comput. Sci. 260 (1-2) (2001) 119–138, Coalgebraic methods in computer science (Lisbon, 1998).
[23] F. Bartels, A. Sokolova, E. de Vink, A hierarchy of probabilistic system types, in: H. P. Gumm, (Ed.), Electronic Notes in Theoretical Computer Science, vol. 82, Elsevier, Amsterdam, 2003.
[24] R.J. Aumann, Borel structures for function spaces, Illinois J. Math. 5 (1961) 614–630.